In this case $u(x, y, z, t) = A(1, 1)\cos t$ is the solution of the problem. The following estimate is obtained for the solution at the point $x = \pi/2, y = \pi/4, z = \pi/8$ for $N = 50\,000, a = 2, b = 1, t = 1$: $\bar{u} = (0.93; 0.91; 0.22)$ (the exact solution is $u = (0.92; 0.88; 0.21)$).

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ON AN ERROR IN THE THEORY OF THE CONFORMAL MAPPING OF SIMILAR REGIONS AND ITS APPLICATION TO THE FLOW PAST A PROFILE*

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The correct value of the peripheral derivative in the conformal mapping of the outsides of similar regions is determined and used in the formula for the velocity distribution over a contour similar to the given profile. The formula contains a correction and examples are given of determining the velocity distribution on an elliptic profile.

When plane fluid flows are investigated, formulas for recomputing the velocity distribution during the passage from the given profile C to a similar profile C_1 (Fig.1) are frequently encountered. The formulas make it possible to alter the hydrodynamic characteristics of a wing. The basic results of this problem are given in /1/, and in all editions of the book /2/.

Let us carry out a critical analysis of the formulas derived, following the account given in /2/. Let the flow pattern past the profile C be known, and the conformal mapping

$$\zeta = F(z, C), \quad F(\infty, C) = \infty \tag{1}$$

be given of the outside of C onto the outside of the unit circle $|\zeta| > 1$, for which, in particular, the correspondence between the points of C and the points of the circumference $\zeta = e^{i\theta}$ ($s = s(\theta)$, s is the arc length along the contour C) is determined.

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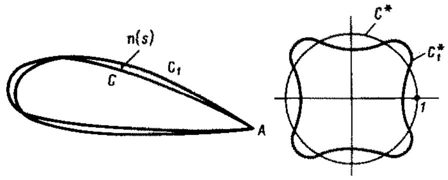

Fig.1

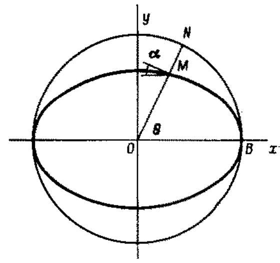

Fig.2

Under the mapping (1) the line C_1 will become the line C_1^*, whose polar equation, apart from terms of the first order of smallness, will have the form

$$\rho = 1 + n\,[s\,(\theta)]\; d\theta/ds = 1 + \delta\,(\theta) \tag{2}$$

where $n\,(s)$ is the section of the normal to the contour C (Fig.1) taken with a definite sign. The mapping of the outside of C_1 onto the outside of the unit circle $|w| > 1$ is given by the function

$$w = F\,(z,\ C_1) = F\,(\zeta,\ C_1^*),\ \zeta = F\,(z,\ C) \tag{3}$$

where $w = F\,(\zeta,\ C_1^*)$ is the mapping of the outside of C_1^* onto $|w| > 1$. From this we obtain, using the formula for differentiating complex functions,

$$|F'\,(z,\ C_1)| = |F'\,(\zeta,\ C_1')|\,|F'\,(z,\ C)| \tag{4}$$

The quantity $|F'\,(z,\ C)|$ on the right-hand side is known, and $|F'\,(\zeta,\ C_1')|$ is found from the theory of conformal mapping of similar regions (/2/, Sect.60) using the expression

$$|F'\,(\zeta,\ C_1^*)| \simeq 1 + \delta\,(\theta) - \frac{1}{4\pi}\int_0^{2\pi} (\delta\,(t) - \delta\,(\theta))\,\sin^{-2}\frac{t-\theta}{2}\,dt \tag{5}$$

The mapping (4) reduces the problem of the flow past the profile C_1 to the problem of the flow past a circular cylinder. Therefore, the magnitude of the velocity at the contour C_1 is found from the formula (/2/, Sect.63, formula (10))

$$|v_1| = |v|\left\{1 + \delta\,(\theta) + \frac{\cos\theta\,\Delta\theta - \cos\theta_0\Delta\theta_0}{\sin\theta - \sin\theta_0} + \frac{1}{2\pi}\int_0^{2\pi}\delta\,(t)\,dt - \right. \tag{6}$$

$$\left.\frac{1}{4\pi}\int_0^{2\pi}(\delta\,(t) - \delta\,(\theta))\,\sin^{-2}\frac{t-\theta}{2}\,dt\right\}$$

$$\Delta\theta = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\theta - t}{2}\,\delta\,(t)\,dt,\quad \Delta\theta_0 = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\theta_0 - t}{2}\,\delta\,(t)\,dt\,.$$

Here $\delta\,(\theta)$ is given by the expression (2), v is the velocity distribution over the contour and $C,\ \theta_0$ is the argument of the image of the point A (Fig.1).

Formula (6) connects the velocities at the points of the contours C_1 and C found on the same normal to C. Below we show that expressions (5) and (6) and their derivation are based on a false assertion.

First we shall test formula (6). Let us choose the profile C in the form of a unit circle, and use an ellipse with the semi-axes $a = 1,\ b = 1 - \varepsilon$ (Fig.2) as the similar profile.

The function characterizing the deviation of the ellipse from a circle along the normal to the latter, is $\delta\,(\theta) = -\varepsilon\sin^2\theta$. Substituting this function into formula (6) in which the velocity of the oncoming flow is taken as unit velocity, and taking into account the fact that we have on the circle $|v| = 2\sin\theta$, we obtain

$$|v_1| = 2\,|\sin\theta|\,[1 - {}^1\!/_2\varepsilon\,(1 - 4\cos 2\theta)]\,. \tag{7}$$

On the other hand, the velocity distribution over the elliptical profile can be found from the "cosine" formula /3/

$$|v_1| = (2 - \varepsilon)\cos\alpha \tag{8}$$

where α is the angle of inclination of the tangent projected on to the ellipse C_1. Remembering that $\cos\alpha = \sin\theta\,(1 + 2\varepsilon\cos^2\theta) + O\,(\varepsilon^2)$, we obtain

$$|v| = 2|\sin\theta|[1 + {}^1/_2\varepsilon(1+2\cos 2\theta)]. \qquad (9)$$

Expression (7) is not identical with (9) and it automatically yields an incorrect value of the maximum velocity at the ellipse $v_{max} = 2 - 5\varepsilon$, while from (8) we obtain, when $\alpha = 0$ (and also from (9)), $v_{max} = 2 - \varepsilon$, i.e. the correction due to the deformation of the contour differs by a factor of 5.

Let us derive formulas (5) and (6). Analysis shows that a formula of the type (5) obtained by mapping the inside of a region bounded by the contour $\rho = 1 - \delta(\theta)$ close to the unit circle onto the inside of a circle is correct (/2/, formula (20)). When the outside of the regions which are almost circular are mapped onto the outside of the unit circle, it is suggested /2/ that the same formula can be used under the condition that the function $\delta(\theta)$ is now determined from the modified equation of the contour

$$\rho = 1 + \delta(\theta). \qquad (10)$$

The approach is jusitified by the assumption that formula (3) (/2/, Sect.60) from which all subsequent relations follow, is valid also for the mapping of the outside of the unit circle with a lune cut-out, onto the outside of the unit circle.

Indeed, formula (3) and a number of subsequent formulas remain valid, taking (10) into account, when the outside regions are mapped. However, formula (18) (Sect.60) in this case becomes incorrect (the sign in the second term must be changed). Removing this error leads to a change of sign in the second term of formulas (5) and (6).

It follows therefore that formulas (6), (7), (10) and (12) in Sect.63 of /2/ and the text in Sect.60 must all be corrected.

We can confirm this even without analysing all relations in Sect.60. Let a nearly circular contour C be given in the z plane (its equation is $r = 1 - \delta(\varphi)$). We define the conformal mapping onto a circle, of a nearly circular region, by the function $w = f(z, C)$. The modulus of the derivative at the points of the boundary is given (/2/, Sect.60, formula 20)) by the formula

$$|f'(z, C)| \simeq 1 + \delta(\varphi) - \frac{1}{4\pi}\int_0^{2\pi}(\delta(t) - \delta(\theta))\sin^{-2}\frac{t-\theta}{2}\,dt. \qquad (11)$$

In order to obtain, in place of (11), the corresponding derivative of the mapping of the outside regions, we shall use the well-known transformations $z = 1/\zeta$ and $w = 1/\omega$. The function $\omega = \omega(\zeta, C) = 1/w(\zeta^{-1}, C)$ describes the mapping of the outside of the contour C defined in the ζ plane, by the equation $\rho = 1 + \delta(\varphi)$, onto the outside of the unit circle in the ω plane. Using the formula for differentiating a complex function, we obtain

$$\left|\frac{d\omega}{d\zeta}\right| = \frac{1}{|w|^2}\left|\frac{dw}{dz}\right|\frac{1}{|\zeta|^2} \quad \left(z = \frac{1}{\zeta}\right). \qquad (12)$$

The quantity $|dw/dz| = |f'(z, C)|$ at the points of the boundary $|w| = 1$ is given by the formula (11) (or (20) in Sect.60), in which $\delta(\varphi)$ is found from the equation $r = 1 - \delta(\varphi)$. Finally, the third multiplier in (12), after expansion in powers of $\delta(\varphi)$, will take the value $1/|\zeta|^2 = 1 - 2\delta(\varphi)$.

As a result we find that we must subtract $2\delta(\theta)$ from expression (5) and the expression within the curly brackets in (6) (and in formula (10), Sect.63 of /2/), and this results in a change of sign in the second term of the above formulas.

In conclusion, we shall correct the test example discussed above. If the quantity $2\delta(\theta) = -2\varepsilon\sin^2\theta$ is subtracted from the expression within the square brackets in (7), we obtain an expression which agrees with (9).

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